Representations of simple finite Lie conformal superalgebras of type W and S

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Abstract

We construct all finite irreducible modules over Lie conformal superalgebras of type W and S.

1 Introduction

Lie conformal superalgebras encode the singular part of the operator product expansion of chiral fields in two-dimensional quantum field theory [7]. On the other hand, they are closely connected to the notion of formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$, that is a Lie superalgebra $\mathfrak{g}$ spanned by the coefficients of a family $\mathcal{F}$ of mutually local formal distributions. Namely, to a Lie conformal superalgebra R one can associate a formal distribution Lie superalgebra (Lie R, R) which establishes an equivalence between the category of Lie conformal superalgebras and the category of equivalence classes of formal distribution Lie superalgebras obtained as quotients of Lie R by irregular ideals [7].

Finite simple Lie conformal algebras were classified in [5] and all their finite irreducible representations were constructed in [3]. According to [5], any finite simple Lie conformal algebra is isomorphic either to the current Lie conformal algebra Cur$\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, or to the Virasoro conformal algebra.

However, the list of finite simple Lie conformal superalgebras is much richer, mainly due to existence of several series of super extensions of the Virasoro conformal algebra. The complete classification of finite simple Lie conformal superalgebras was obtained in [6]. The list consists of current Lie conformal superalgebras Cur$\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra, four series of "Virasoro like" Lie conformal superalgebras W_n $(n \geq 0)$, $S_{n,b}$ and $\widetilde{S}_n$ $(n \geq 2, b \in \mathbb{C})$, K_n $(n \geq 0)$, and the exceptional Lie conformal superalgebra CK_6.

All finite irreducible representations of the simple Lie conformal superalgebras Cur$\mathfrak{g}$, $K_0 = Vir$ and K_1 were constructed in [3], and those of $S_{2,0}$, $W_1 = K_2$, K_3, K_4 in [4].

The main result of the present paper is the construction of all finite irreducible modules over the Lie conformal superalgebras W_n, $S_{n,b}$ and $\widetilde{S}_n$. The proof relies on the method developed in [3], that is, the observation that representation theory of Lie conformal superalgebras is controlled by the representation theory of the (extended) annihilation superalgebra, and a lemma from [2]. In our cases, this reduces to the study of certain representations of the Lie superalgebra $W(1,n)_+$ of all vector fields on a superline (an affine superspace of dimension $(1|n)$) and the Lie superalgebra $S(1,n)_+$ of such vector fields with zero divergence. As in [8], [9], we follow the approach developed for representations of infinite-dimensional simple linearly compact Lie algebras by A. Rudakov in [10]. The problem reduces to the description of the so called degenerate modules, and for the later we have to study singular vectors.

The paper is organized as follows. In Section 2, we recall the notions and some basic facts on of formal distributions, Lie conformal superalgebras and their modules. In Section 3, we recall some simple facts of the representation theory of infinite-dimensional simple linearly compact Lie superalgebras. In Section 4, we describe the conformal Lie superalgebra W_n and we classify its finite irreducible conformal modules by studying the corresponding singular vectors. In Section 5, we obtain similar results for the Lie conformal superalgebra $S_n = S_{n,0}$. Finally, in Section 6, we complete the cases $S_{n,b}$ and $\widetilde{S}_n$. In all cases (as in [10]) the answer has a geometric meaning: all finite irreducible modules are either "non-degenerate" tensor modules, or occur as cokernels of the differential in the conformal de Rham complex, or are duals of the latter.

Note that similar results for arbitrary non-super Lie pseudoalgebras of types W and S have been obtained in [11].

The remaining cases of the Lie conformal superalgebras K_n and CK_6 will be worked out in the subsequent publication.

2 Formal distributions, Lie conformal superalgebras and their modules

First we introduce the basic definitions and notations, see [7, 5]. Let $\mathfrak{g}$ be a Lie superalgebra. A $\mathfrak{g}$-valued *formal distribution* in one indeterminate z is a series in the indeterminate z

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad a_n \in \mathfrak{g}.$$

The vector superspace of all formal distributions, $\mathfrak{g}[[z, z^{-1}]]$, has a natural structure of a $\mathbb{C}[\partial_z]$-module. We define

$$\mathrm{Res}_z a(z) = a_0.$$

Let $a(z), b(z)$ be two $\mathfrak{g}$-valued formal distributions. They are called *local* if

$$(z-w)^N [a(z), b(w)] = 0 \qquad \text{for} \quad N >> 0.$$

Let $\mathfrak{g}$ be a Lie superalgebra, a family $\mathcal{F}$ of $\mathfrak{g}$-valued formal distributions is called a *local family* if all pairs of formal distributions from $\mathcal{F}$ are local. Then, the pair $(\mathfrak{g}, \mathcal{F})$ is called a *formal distribution Lie superalgebra* if $\mathcal{F}$ is a local family of $\mathfrak{g}$-valued formal distributions and $\mathfrak{g}$ is spanned by the coefficients of all formal distributions in $\mathcal{F}$. We define the *formal δ-function* by

$$\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z}\right)^n .$$

Then it is easy to show ([7], Corollary 2.2)), that two local formal distributions are local if and only if the bracket can be represented as a finite sum of the form

$$[a(z), b(w)] = \sum_j [a(z)_{(j)} b(w)] \ \partial_w^j \delta(z-w)/j!,$$

where $[a(z)_{(j)} b(w)] = \mathrm{Res}_z (z-w)^j [a(z), b(w)]$. This is called the *operator product expansion*. Then we obtain a family of operations ${}_{(n)}$, $n \in \mathbb{Z}_+$, on the space of formal distributions. By taking the generating series of these operations, we define the λ-bracket:

$$[a_\lambda b] = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!} [a_{(n)} b].$$

The properties of the λ-bracket motivate the following definition:

Definition 2.1. A *Lie conformal superalgebra* R is a left $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map,

$$R \otimes R \longrightarrow \mathbb{C}[\lambda] \otimes R, \qquad a \otimes b \mapsto a_\lambda b$$

called the λ-bracket, and satisfying the following axioms (a, b, $c \in R$),

Conformal sesquilinearity $\quad [\partial a_\lambda b] = -\lambda [a_\lambda b], \qquad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b],$

Skew-symmetry $\quad [a_\lambda b] = -(-1)^{p(a)p(b)} [b_{-\lambda-\partial}\ a],$

Jacobi identity $\quad [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)} [b_\mu [a_\lambda c]].$

Here and further $p(a) \in \mathbb{Z}/2\mathbb{Z}$ is the parity of a.

A Lie conformal superalgebra is called *finite* if it has finite rank as a $\mathbb{C}[\partial]$-module. The notions of homomorphism, ideal and subalgebras of a Lie conformal superalgebra are defined in the usual way. A Lie conformal superalgebra R is *simple* if $[R_\lambda R] \neq 0$ and contains no ideals except for zero and itself.

Given a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$ denote by $\bar{\mathcal{F}}$ the minimal subspace of $\mathfrak{g}[[z, z^{-1}]]$ which contains $\mathcal{F}$ and is closed under all j-th products and invariant under ∂_z. Due to Dong's lemma, we know that $\bar{\mathcal{F}}$ is a local family as well. Then $\mathrm{Conf}(\mathfrak{g}, \mathcal{F}) := \bar{\mathcal{F}}$ is the Lie conformal superalgebra associated to the formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$.

In order to give the (more or less) reverse functorial construction, we need the notion of *affinization* $\widetilde{R}$ of a conformal algebra R (which is a generalization of that for vertex algebras [1]). We let $\widetilde{R} = R[t, t^{-1}]$ with $\widetilde{\partial} = \partial + \partial_t$ and the λ-bracket [7]:

$$[af(t)_\lambda bg(t)] = [a_{\lambda+\partial_t} b] f(t) g(t')|_{t'=t}. \tag{2.1}$$

The 0-th product is:

$$[at^n_{(0)} bt^m] = \sum_{j \in \mathbb{Z}_+} \left(\begin{array}{c} m \\ j \end{array} \right) [a_j b] t^{m+n-j}. \tag{2.2}$$

Observe that $\widetilde{\partial}\widetilde{R}$ is an ideal of $\widetilde{R}$ with respect to the 0-th product. We let $\mathrm{Alg}R = \widetilde{R}/\widetilde{\partial}\widetilde{R}$ with the 0-th product and let

$$\mathcal{R} = \{\sum_{n \in \mathbb{Z}} (at^n) z^{-n-1} = a\delta(t-z) \mid a \in R\}.$$

Then $(\mathrm{Alg}R, \mathcal{R})$ is a formal distribution Lie superalgebra. Note that Alg is a functor from the category of Lie conformal superalgebras to the category of formal distribution Lie superalgebras. On has [7]:

$$\mathrm{Conf}(\mathrm{Alg}R) = R, \quad \mathrm{Alg}(\mathrm{Conf}(\mathfrak{g}, \mathcal{F})) = (\mathrm{Alg}\bar{\mathcal{F}}, \bar{\mathcal{F}}).$$

Note also that $(\mathrm{Alg}R, \mathcal{R})$ is the *maximal formal distribution superalgebra* associated to the conformal superalgebra R, in the sense that all formal distribution Lie superalgebras $(\mathfrak{g}, \mathcal{F})$ with $\mathrm{Conf}(\mathfrak{g}, \mathcal{F}) = R$ are quotients of $(\mathrm{Alg}R, \mathcal{R})$ by irregular ideals (that is, an ideal I in $\mathfrak{g}$ with no non-zero $b(z) \in \mathcal{R}$ such that $b_n \in I$). Such formal distribution Lie superalgebras are called *equivalent*.

We thus have an equivalence of categories of conformal Lie superalgebras and equivalence classes of formal distribution Lie superalgebras. So the study of formal distribution Lie superalgebras reduces to the study of conformal Lie superalgebras.

An important tool for the study of Lie conformal superalgebras and their modules is the (extended) annihilation algebra. The *annihilation algebra* of a Lie conformal superalgebra R is the subalgebra $\mathcal{A}(R)$ (also denoted by $(\mathrm{Alg}R)_+$) of the Lie superalgebra $\mathrm{Alg}R$ spanned by all elements at^n, where $a \in R, n \in \mathbb{Z}_+$. It is clear from (2.2) that this is a subalgebra, which is invariant with respect to the derivation $\partial = -\partial_t$ of $\mathrm{Alg}R$. The *extended annihilation algebra* is defined as

$$\mathcal{A}(R)^e = (\mathrm{Alg}R)^+ := \mathbb{C}\partial \ltimes (\mathrm{Alg}R)_+.$$

Introducing the generating series

$$a_\lambda = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} (at^j), \tag{2.3}$$

we obtain from (2.2):

$$[a_\lambda, b_\mu] = [a_\lambda b]_{\lambda+\mu}, \quad \partial(a_\lambda) = (\partial a_\lambda) = \lambda(a_\lambda). \tag{2.4}$$

Now let $\mathfrak{g}$ be a Lie superalgebra, and let V be a $\mathfrak{g}$-module. Given a $\mathfrak{g}$-valued formal distribution $a(z)$ and a V-valued formal distribution $v(z)$ we may consider the formal distribution $a(z)v(w)$ and the pair $(a(z), v(z))$ is called *local* if $(z-w)^N(a(z)v(w)) = 0$ for $N >> 0$. As before, we have an expansion of the form:

$$a(z)v(w) = \sum_j \Big(a(z)_{(j)}v(w)\Big)\ \partial_w^j \delta(z-w)/j!,$$

where $a(w)_{(j)}v(w) = \mathrm{Res}_z(z-w)^j a(z)v(w)$ and the sum is finite. By taking the generating series of these operations, we define the λ-*action of* $\mathfrak{g}$ *on* V:

$$a(w)_\lambda v(w) = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}\Big(a(w)_{(n)}v(w)\Big), \qquad \text{(finite sum)}.$$

It has the following properties:

$$\partial_z a(z)_\lambda v(z) = -\lambda a(z)_\lambda v(z), \qquad a(z)_\lambda \partial_z v(z) = (\partial_z + \lambda)(a(z)_\lambda v(z)),$$

and

$$[a(z)_\lambda, b(z)_\mu]v(z) = [a(z)_\lambda b(z)]_{\lambda+\mu} v(z).$$

This motivate the following definition:

Definition 2.2. A *module* M over a Lie conformal superalgebra R is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R\otimes M \longrightarrow \mathbb{C}[\lambda]\otimes M$, $a\otimes v \mapsto a_\lambda v$, satisfying the following axioms $(a,\ b \in R),\ v\in M$,

$(M1)_\lambda \qquad (\partial a)^M_\lambda v = [\partial^M, a^M_\lambda]v = -\lambda a^M_\lambda v,$

$(M2)_\lambda \qquad [a^M_\lambda, b^M_\mu]v = [a_\lambda b]^M_{\lambda+\mu} v.$

An R-module M is called *finite* if it is finitely generated over $\mathbb{C}[\partial]$. An R-module M is called *irreducible* if it contains no non-trivial submodule, where the notion of submodule is the usual one.

As before, if $\mathcal{F} \subset \mathfrak{g}[[z,z^{-1}]]$ is a local family and $\mathcal{E} \subset V[[z,z^{-1}]]$ is such that all pairs $(a(z), v(z))$, where $a(z)\in\mathcal{F}$ and $v(z)\in\mathcal{E}$, are local, let $\bar{\mathcal{E}}$ be the minimal subspace of $V[[z,z^{-1}]]$ which contains E and all $a(z)_{(j)}v(z)$ for $a(z)\in\mathcal{F}$ and $v(z)\in\mathcal{E}$, and is ∂_z-invariant. Then it is easy to show that all pairs $(a(z), v(z))$, where $a(z)\in\bar{\mathcal{F}}$ and $v(z)\in\bar{\mathcal{E}}$, are local and $a(z)_{(j)}(\bar{\mathcal{E}}) \subset \bar{\mathcal{E}}$ for all $a(z)\in\bar{\mathcal{F}}$.

Let $\mathcal{F}$ be a local family that spans $\mathfrak{g}$ and let $\mathcal{E}\subset V[[z,z^{-1}]]$ be a family that span V. Then $(V,\mathcal{E})$ is called a *formal distribution module* over the formal distribution Lie superalgebra $(\mathfrak{g},\mathcal{F})$ if all pairs $(a(z), v(z))$, where $a(z)\in\mathcal{F}$ and $v(z)\in\mathcal{E}$, are local. It follows that a formal distribution module $(V,\mathcal{E})$ over a

formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$ give rise to a module $\mathrm{Conf}(V, \mathcal{E}) := \bar{\mathcal{E}}$ over the conformal Lie superalgebra $\mathrm{Conf}(\mathfrak{g}, \mathcal{F})$.

In the same way as above, we have an equivalence of categories of modules over a Lie conformal superalgebra R and equivalence classes or formal distribution modules over the Lie superalgebra $\mathrm{Alg}R$. Namely, given an R-module M, one defines its *affinization* $\widetilde{M} = M[t, t^{-1}]$ as a $\widetilde{R}$-module with $\widetilde{\partial} = \partial + \partial_t$ and the λ-action similar to (2.1):

$$af(t)_\lambda vg(t) = (a_{\lambda+\partial_t} v) f(t) g(t')|_{t'=t}. \tag{2.5}$$

The 0-th action is:

$$at^n_{(0)} vt^m = \sum_{j \in \mathbb{Z}_+} \left(\begin{array}{c} m \\ j \end{array} \right) (a_j v) t^{m+n-j}. \tag{2.6}$$

Observe that $\widetilde{\partial}\widetilde{M}$ is invariant with respect to the 0-th action and $(\widetilde{\partial}\widetilde{R})_{(0)}\widetilde{M} = 0$, hence the 0-th action of $\widetilde{R}$ on $\widetilde{M}$ induces a representation of the Lie superalgebra $\mathrm{Alg}R = \widetilde{R}/\widetilde{\partial}\widetilde{R}$ on $V(M) := \widetilde{M}/\widetilde{\partial}\widetilde{M}$. Let $\mathcal{M} = \{v\delta(z-t) | v \in M\}$. Then $(V(M), \mathcal{M})$ is a formal distribution module over the formal distribution Lie superalgebra $(\mathrm{Alg}R, \mathcal{R})$, which is maximal in the sense that all conformal $(\mathrm{Alg}R, \mathcal{R})$ modules $(V, \mathcal{E})$ such that $\bar{\mathcal{E}} \simeq M$ as R-modules are quotients of $(V(M), \mathcal{M})$ by irregular submodules. Such formal distribution modules are called equivalent, and we get an equivalence of categories of R-modules and equivalence classes of formal distribution $(\mathrm{Alg}R, \mathcal{R})$-modules.

Formula (2.4) implies the following important proposition relating modules over a Lie conformal superalgebra R to continuous modules over the corresponding extended annihilation Lie superalgebra $(\mathrm{Alg}R)^+$.

Proposition 2.3. *[3] A module over a Lie conformal superalgebra R is the same as a continuous module over the Lie superalgebra $(\mathrm{Alg}R)^+$, i.e. it is an $(\mathrm{Alg}R)^+$-module satisfying the property*

$$a_\lambda m \in \mathbb{C}[\lambda] \otimes M \quad \textit{for any } a \in R, m \in M. \tag{2.7}$$

(One just views the action of the generating series a_λ of $(\mathrm{Alg}R)^+$ as the λ-action of $a \in R$).

Denote by $V(M)_+$ the span of elements $\{vt^n | v \in M, n \in \mathbb{Z}_+\}$ in $V(M)$. It is clear from (2.5) that $V(M)^+$ is an $(\mathrm{Alg}R)^+$ submodule, hence an R -module by Proposition 2.3. We denote by $V(M)^*_+$ the restricted dual of $V(M)_+$, i.e. the space of all linear functions on $V(M)_+$ which vanish on all but finite number of subspaces Mt^n, with $n \in \mathbb{Z}_+$. This is an $(\mathrm{Alg}R)^+$-module and hence an R-module as well. The *conformal dual* M^* to an R-module M is defined as

$$M^* = \{f_\lambda : M \to \mathbb{C}[\lambda] \mid f_\lambda(\partial m) = \lambda f_\lambda(m)\},$$

with the structure of $\mathbb{C}[\partial]$-module $(\partial f)_\lambda(m) = -\lambda f_\lambda(m)$, with the following λ-action of R:

$$(a_\lambda f)_\mu(m) = -(-1)^{p(a)(p(f)+1)} f_{\mu-\lambda}(a_\lambda m), \quad a \in R, m \in M.$$

Given a homomorphism of conformal R-modules $T : M \to N$, we define the transpose homomorphism $T^* : N^* \to M^*$ by

$$[T^*(f)]_\lambda(m) = -f_\lambda(T(m))$$

Proposition 2.4. *Let $T : M \to N$ be an injective homomorphism of R-modules such that $N/Im\, T$ is finitely generated torsion-free $\mathbb{C}[\partial]$-module. Then T^* is surjective.*

Proof. Since $N/\mathrm{Im}\, T$ is finitely generated torsion-free, then it is free and therefore a proyective $\mathbb{C}[\partial]$-module. Hence, the short exact sequence $0 \to \mathrm{Im}\, T \to N \to N/\mathrm{Im}\, T \to 0$ is split and $N = \mathrm{Im}\, T \oplus L$ as $\mathbb{C}[\partial]$-module. Now, given $\alpha \in M^*$, we define $\beta \in N^*$ as follows

$$\beta_\lambda(T(m)) = \alpha_\lambda(m), \quad m \in M, \qquad \beta_\lambda(l) = 0, \quad l \in L.$$

Then β is well-defined since T is injective and β belong to N^* since L is a complementary $\mathbb{C}[\partial]$-submodule, finishing the proof. □

Remark 2.5. Observe that the injectivity is not enough (cf. Remark 4.12). Namely, let $R = Vir = \mathbb{C}[\partial]L$ the Virasoro conformal algebra with λ-bracket $[L_\lambda L] = (2\lambda + \partial)L$. Consider the following Vir-modules:

$$\Omega_0 = \mathbb{C}[\partial]m, \;\; \text{with } L_\lambda m = (\lambda + \partial)m; \qquad \Omega_1 = \mathbb{C}[\partial]n, \;\; \text{with } L_\lambda n = \partial n.$$

Then it is easy to see that the map $d : \Omega_0 \to \Omega_1$ given by $d(m) = \partial n$ is an injective homomorphism of R-modules, but the dual map $d^* : \Omega_1^* \to \Omega_0^*$ given by $d^*(m^*) = \partial n^*$ is not surjective.

Proposition 2.6. *Let $T : M \to N$ be a homomorphism of R-modules such that $N/Im\, T$ is finitely generated torsion-free $\mathbb{C}[\partial]$-module. Then the standard map $\psi : N^*/Ker\, T^* \to (M/Ker\, T)^*$, given by $[\psi(\bar{f})]_\lambda(\bar{m}) = f_\lambda(T(m))$ (where by the bar we denote the corresponding class in the quotient) is an isomorphism of R-modules.*

Proof. Using Proposition 2.4 the proof follows by standard arguments. □

Proposition 2.7. *If M is an R-module finitely generated (over $\mathbb{C}[\partial]$), then $M^{**} \simeq M$.*

Proof. Let $M = \oplus\, \mathbb{C}[\partial]m_i$ (finite sum), with $a_\lambda m_j = \sum_k P_{jk}(\lambda, \partial)m_k$. Then $M^* = \oplus\, \mathbb{C}[\partial]m_i^*$, with $(m_i^*)_\lambda(m_k) = \delta_{i,k}$ and

$$(a_\lambda m_i^*)_\mu(m_j) = -(m_i^*)_{\mu-\lambda}(a_\lambda m_j) = -\sum_k (m_i^*)_{\mu-\lambda}(P_{jk}(\lambda, \partial)m_k) = P_{ji}(\lambda, \mu-\lambda).$$

Therefore,

$$(a_\lambda m_i^*) = -\sum_j P_{ji}(\lambda, -\partial - \lambda)m_j^*,$$

and the last formula shows that by taking the dual again we obtain

$$(a_\lambda m_i^{**}) = \sum_j P_{ij}(\lambda, \partial) m_j^{**}.$$

Hence the map $m_i \mapsto m_i^{**}$ gives us the isomorphism between M and M^{**}. $\square$

Proposition 2.8. *(a) The map $M \to V(M)/V(M)_+$ given by $v \mapsto vt^{-1}$ mod $V(M)_+$ is an isomorphism of $(\mathrm{Alg}R)^+$- (and R-)modules.*
(b)The map $V(M)_+^* \to M^*$ defined by $f \mapsto f_\lambda$, where

$$f_\lambda(m) = \sum_{j \in \mathbb{Z}_+} \frac{(-\lambda)^j}{j!} f(mt^j),$$

is an isomorphism of $(\mathrm{Alg}R)^+$- (and R-)modules.

Proof. A direct verification. $\square$

Assuming that R is finite, choose a finite set of generators of this $\mathbb{C}[\partial]$-module: $\{a^i | i \in I\}$, and for each $m \in \mathbb{Z}_+$, denote by $(\mathrm{Alg}R)^+_{(m)}$ the $\mathbb{C}$-span of all elements $a^i t^j, i \in I, j \geq m$ of $(\mathrm{Alg}R)^+$. This defines a descending filtration of $(\mathrm{Alg}R)^+$ by subspaces of finite codimension:

$$(\mathrm{Alg}R)^+ \supset (\mathrm{Alg}R)_+ = (\mathrm{Alg}R)_{(0)} \supset (\mathrm{Alg}R)_{(1)} \supset \dots . \tag{2.8}$$

It is easy to see from (2.2) that there exists $s \in \mathbb{Z}_+$ such that for all $k, r \in \mathbb{Z}_+$ one has:

$$[(\mathrm{Alg}R)_{(k)}, (\mathrm{Alg}R)_{(r)}] \subset (\mathrm{Alg}R)_{(k+r-s)}. \tag{2.9}$$

In particular, $(\mathrm{Alg}R)_r := (\mathrm{Alg}R)_{(r+s)}$ is a filtration of $(\mathrm{Alg}R)^+$ by subalgebras of finite codimension.

Given an R-module M, it is an $(\mathrm{Alg}R)^+$-module (by Proposition 2.3), and we let for $j \in \mathbb{Z}_+$:

$$M_{(j)} = \{v \in M | (\mathrm{Alg}R)_{(j)} v = 0\}. \tag{2.10}$$

The subspaces $M_{(j)}$ form an ascending filtration of M by $(\mathrm{Alg}R)_0$-invariant subspaces. The following proposition is a special case of Lemma 14.4 from [2].

Proposition 2.9. *let R be a finite Lie conformal superalgebra and let M be a finite R-module such that*

$$M^R = \{m \in M | R_\lambda m = 0\} (= M_{(0)})$$

is finite dimensional (over $\mathbb{C}$). Then all subspaces $M_{(j)}$ are finite-dimensional. In particular M is locally finite as an $(\mathrm{Alg}R)_0$-module, meaning that any $m \in M$ is contained in a finite-dimensional $(\mathrm{Alg}R)_0$-invariant subspace.

This Proposition together with the results of the following section will provide a characterization of all finite irreducible modules over a finite Lie conformal superalgebra in terms of certain (quotients of) induced modules over the extended annihilation algebra.

3 General remarks on representations of linearly compact Lie superalgebras

We follow the approach developed for representations of infinite-dimensional simple linearly compact Lie algebras by A. Rudakov in [10]. In this section we will follow [8].

We shall consider continuous representations in spaces with discrete topology. The continuity of a representation of a linearly compact Lie superalgebra L in a vector space V with discrete topology means that the stabilizer $L_v = \{g \in L | gv = 0\}$ of any $v \in V$ is an open (hence of finite codimension) subalgebra of L.

Let L be a simple linearly compact Lie superalgebra. In some cases (the examples studied in the following sections), L has a $\mathbb{Z}$-gradation of the form

$$L = \oplus_{m \geq -1} L_m, \tag{3.1}$$

this gives a triangular decomposition

$$L = L_- \oplus L_0 \oplus L_+, \qquad \text{with} \qquad L_\pm = \oplus_{\pm m > 0} L_m. \tag{3.2}$$

Let $L_{\geq 0} = L_{>0} \oplus L_0$. Denote by $P(L, L_{\geq 0})$ the category of all continuous L-modules V, where V is a vector space with discrete topology, that are $L_{\geq 0}$-locally finite, that is any $v \in V$ is contained in a finite-dimensional $L_{\geq 0}$-invariant subspace. When talking about representations of L, we shall always mean modules from $P(L, L_{\geq 0})$. Modules in this category are called *finite continuous L-modules*.

In general, in most (but not all cases) of simple L, by taking $L_{\geq 0}$ certain maximal open subalgebra, one can choose L_- to be a subalgebra. Taking an ordered basis of L_-, we denote by $U(L_-)$ the span of all PBW monomials in this basis. We have $U(L) = U(L_-) \otimes U(L_{\geq 0})$, as vector spaces (here and further $U(L)$ stands for the universal enveloping algebra of the Lie superalgebra L). It follows that any irreducible L-module V in the category $P(L, L_{\geq 0})$ is finitely generated over $U(L_-)$:

$$V = U(L_-)E$$

for some finite-dimensional subspace E. This property is very important in the theory of conformal modules [3].

Given an $L_{\geq 0}$-module F, we may consider the associated induced L-module

$$M(F) = \operatorname{Ind}_{L_{\geq 0}}^{L} F = U(L) \otimes_{U(L_{\geq 0})} F,$$

called the *generalized Verma module* associated to F. Sometimes, we shall omit L and $L_{\geq 0}$, and simply denote it as $\operatorname{Ind} F$.

Let V be an L-module. The elements of the subspace

$$\operatorname{Sing}(V) := \{v \in V | L_{>0} v = 0\}$$

are called *singular vectors*. For us the most important case is when $V = M(F)$. The $L_{\geq 0}$-module F is canonically an $L_{\geq 0}$-submodule of $M(F)$, and $\operatorname{Sing}(F)$ is a

subspace of $\mathrm{Sing}(M(F))$, called the *subspace of trivial singular vectors*. Observe that $M(F) = F \oplus F_+$, where $F_+ = U_+(L_-) \otimes F$ and $U_+(L_-)$ is the augmentation ideal in the symmetric algebra $U(L_-)$. Then

$$\mathrm{Sing}_+(M(F)) := \ \mathrm{Sing}(M(F)) \cap F_+$$

are called the *non-trivial singular vectors*.

Theorem 3.1. *[8][10] (a) If F is a finite-dimensional $L_{\geq 0}$-module, then $M(F)$ is in $P(L, L_{\geq 0})$.*

(b) In any irreducible finite-dimensional $L_{\geq 0}$-module F the subalgebra L_+ acts trivially.

(c) If F is an irreducible finite-dimensional $L_{\geq 0}$-module, then $M(F)$ has a unique maximal submodule.

(d) Denote by I(F) the quotient by the unique maximal submodule of $M(F)$. Then the map $F \mapsto I(F)$ defines a bijective correspondence between irreducible finite-dimensional L_0-modules and irreducible L-modules in $P(L, L_{\geq 0})$, the inverse map being $V \mapsto Sing(V)$.

(e) An L-module $M(F)$ is irreducible if and only if the L_0-module F is irreducible and $M(F)$ has no non-trivial singular vectors.

Remark 3.2. The correspondence defined in Theorem 3.1(d) provides the classification of irreducible modules of the category $P(L, L_{\geq 0})$. Also, we would like to remark that in general $\mathrm{Sing}_+(M(F))$ generates a proper submodule in the L-module $M(F)$, but the factor by this submodule is not necessarily irreducible, there could appear new non-trivial singular vectors. However this happens very rarely (see [9] for an example and cf. Remark 4.8) and in most cases it can be proven that the factor module will be irreducible.

4 Lie conformal superalgebra W_n and its finite irreducible modules

4.1 Definition of W_n and the induced modules

According to [5], any finite simple Lie conformal algebra is isomorphic either to $\mathrm{Cur}\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, or to the Virasoro conformal algebra.

However, the list of finite simple Lie conformal superalgebras is much richer, mainly due to existence of several series of super extensions of the Virasoro conformal algebra, see [6].

The first series is associated to the Lie superalgebra $W(1, n)$ $(n \geq 1)$. More precisely, let $\Lambda(n)$ be the Grassmann superalgebra in the n odd indeterminates $\xi_1, \xi_2, \ldots, \xi_n$. Set $\Lambda(1, n) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$, then

$$W(1, n) = \{a\partial_t + \sum_{i=1}^{n} a_i\partial_i | a, a_i \in \Lambda(1, n)\}, \tag{4.1}$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$ and $\partial_t = \frac{\partial}{\partial t}$ are odd and even derivations respectively. Then $W(1,n)$ is a formal distribution Lie superalgebra with spanning family of (pairwise local) formal distributions:

$$\mathcal{F} = \{\delta(t-z)a \mid a \in W(n)\} \cup \{\delta(t-z)f\partial_t \mid f \in \Lambda(n)\}.$$

where $W(n) = \{\sum_{i=1}^n a_i\partial_i | a_i \in \Lambda(n)\}$ is the (finite-dimensional) Lie superalgebra of all derivations of $\Lambda(n)$. The associated Lie conformal superalgebra W_n is defined as

$$W_n = \mathbb{C}[\partial] \otimes (W(n) \oplus \Lambda(n)). \tag{4.2}$$

The λ-bracket is defined as follows ($a, b \in W(n); f, g \in \Lambda(n)$):

$$[a_\lambda b] = [a, b], \quad [a_\lambda f] = a(f) - (-1)^{p(a)p(f)}\lambda f a, \quad [f_\lambda g] = -(\partial + 2\lambda)fg \tag{4.3}$$

The Lie conformal algebra W_n is simple for $n \geq 0$ and has rank $(n+1)2^n$.

The annihilation subalgebra is

$$\mathcal{A}(W_n) = W(1,n)_+ = \{a\partial_t + \sum_{i=1}^n a_i\partial_i | a, a_i \in \Lambda(1,n)_+\}, \tag{4.4}$$

where $\Lambda(1,n)_+ = \mathbb{C}[t] \otimes \Lambda(n)$. The extended annihilation subalgebra is

$$\mathcal{A}(W_n)^e = W(1,n)^+ = \mathbb{C}\partial_t \ltimes W(1,n)_+,$$

and therefore it is isomorphic to the direct sum of $W(1,n)_+$ and a commutative 1-dimensional Lie algebra.

The $\mathbb{Z}$-gradation in (3.1) is obtained by letting

$$\deg t = \deg \xi_i = 1 = -\deg \partial_t = -\deg \partial_i.$$

If $L = W(1,n)_+$, then $L_{-1} =< \partial_t, \partial_1, \ldots, \partial_n >$, where ∂_t is an even element and $\partial_1, \ldots, \partial_n$ are odd elements of a basis in L_{-1}. Note also that $L_0 \simeq gl(1|n)$.

From now on, we shall use the notation $\partial_0 = \partial_t$. Explicitly, we have

$$L_0 =< \{t\partial_i, \xi_i\partial_j \ : \ 0 \leq i,j \leq n\} > .$$

In order to write explicitly weights for vectors in $W(1,n)_+$-modules, we would consider the basis

$$t\partial_0; t\partial_0 + \xi_1\partial_1, \ldots, t\partial_0 + \xi_n\partial_n$$

for the Cartan subalgebra H in $W(1,n)_+$, and we write the weight of an eigenvector for the Cartan subalgebra H as a tuple

$$\bar{\mu} = (\mu; \lambda_1, \ldots, \lambda_n)$$

for the corresponding eigenvalues of the basis.

4.2 Modules of Laurent differential forms

4.2.1 *Restricted dual.* Our algebra $L = W(1,n)_+$, and in the last section $S(1,n)_+$, are $\mathbb{Z}$-graded (super)algebras and the modules we intend to study are graded modules, i.e. an L-module V is a direct sum $V = \oplus_{m\in\mathbb{Z}} V_m$ of finite-dimensional subspaces V_m, and $L_k \cdot V_m \subset V_{k+m}$. For a graded module V we define the *restricted dual module* $V^\#$ as

$$V^\# = \oplus_{m\in\mathbb{Z}} (V_m)^*.$$

hence $V^\#$ is a subspace of V^* and it is invariant with respect to the contragradient action, so it defines an L-module structure. Observe that $(V^\#)^\# = V$.

In our situation, we have $L_{-1} = \langle \partial_0, \partial_1, \dots, \partial_n \rangle$, then any L-module become a $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_n]$-module. Hence, a module V is a free $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_n]$-module if and only if $V^\#$ is a cofree module, i.e. it is isomorphic to a direct sum of copies of the standard module $\mathbb{C}[z, \rho_1, \dots, \rho_n]$, with $\partial_0 \cdot f = \frac{\partial}{\partial z} f$, and $\partial_i \cdot f = \frac{\partial}{\partial \rho_i} f$.

An induced module $\mathrm{Ind}_{L_{\geq 0}}^{L} F$ is by definition a free $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_n]$-module, so the co-induced (or produced) module

$$\mathrm{Cnd} F^\# = (\mathrm{Ind} F)^\#$$

will be cofree.

4.2.2 *Differential forms modules.* In order to define the differential forms one considers an odd variable dt and even variables $d\xi_1, \dots, d\xi_n$ and defines the differential forms to be the (super)commutative algebra freely generated by these variables over $\Lambda(1,n)_+ = \mathbb{C}[t] \otimes \Lambda(n)$, or

$$\Omega_+ = \Lambda(1,n)_+[d\xi_1, \dots, d\xi_n] \otimes \Lambda[dt].$$

Generally speaking Ω_+ is just a polynomial (super)algebra over a big set of variables

$$t, \xi_1, \dots, \xi_n, dt, d\xi_1, \dots, d\xi_n,$$

where the parity is

$$p(t) = 0, \quad p(\xi_i) = 1, \quad p(dt) = 1, \quad p(d\xi_i) = 0.$$

These are called *(polynomial) differential forms*, and we define the *Laurent differential forms* to be the same algebra over $\Lambda(1,n) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$:

$$\Omega = \Lambda(1,n)[d\xi_1, \dots, d\xi_n] \otimes \Lambda[dt].$$

We would like to consider a fixed complementary subspace Ω_- to Ω_+ in Ω chosen as follows

$$\Omega_- = t^{-1}\mathbb{C}[t^{-1}] \otimes \Lambda(n) \otimes \mathbb{C}[d\xi_1, \dots, d\xi_n] \otimes \Lambda[dt].$$

For the differential forms we need the usual differential degree that measure only the involvement of the differential variables $dt, d\xi_1, \ldots, d\xi_n$, that is

$$\deg\, t = 0,\ \deg\, \xi_i = 0,\ \deg\, dt = 1,\ \deg\, d\xi_i = 1.$$

As a result, the degree of a function is zero an it gives us the *standard* $\mathbb{Z}$-*gradation* both on Ω and $\Omega_\pm$. As usual, we denote by $\Omega^k, \Omega^k_\pm$ the corresponding graded components.

We denote by Ω^k_c the special subspace of differential forms with constant coefficients in Ω_k.

The operator d is defined on Ω as usual by the rules $d \cdot t = dt, d \cdot \xi_i = d\xi_i, d \cdot d\xi_i = 0$, and the identity

$$d(fg) = (df)g + (-1)^{p(f)} f dg,$$

Observe that d maps both Ω_+ and Ω_- into themselves.

As usual, we extend the natural action of $W(1,n)_+$ on $\Lambda(1,n)$ to the hole Ω by imposing the property

$$D \cdot d = (-1)^{p(D)} d \cdot D, \qquad D \in W(1,n)_+,$$

that is, D (super)commutes with d. It is clear that Ω_+ and all the subspaces Ω^k are invariant. Hence Ω^k_+ and Ω^k are $W(1,n)_+$-modules, which are called the *natural representations* of $W(1,n)_+$ in differential forms.

We define the action of $W(1,n)_+$ on Ω_- via the isomorphism of Ω_- with the factor of Ω by Ω_+. Practically this means that in order to compute $D \cdot f$, where $f \in \Omega_-$, we apply D to f and "disregard terms with non-negative powers of t".

The operator d restricted to $\Omega^k_\pm$ defines an odd morphism between the corresponding representations. Clearly the image and the kernel of such a morphism are submodules in $\Omega^k_\pm$.

Let $\Theta^k_c = (\Omega^k_c)^\#$ and $\Theta^k_+ = (\Omega^k_+)^\#$. In the rest of this subsection, we consider $L = W(1,n)_+$.

Proposition 4.1. *(1) The L_0-module $\Theta^k_c, k \geq 0$ is irreducible with highest weight*

$$(0; 0, \ldots, 0, -k), \quad k \geq 0.$$

(2) The L-module $\Theta^k_+,\ k \geq 0$ contains Θ^k_c and this inclusion induces the isomorphism

$$\Theta^k_+ = \mathrm{Ind}\ \Theta^k_c.$$

(3) The dual maps $d^\# : \Theta^{k+1}_+ \to \Theta^k_+$ are morphisms of L-modules. The kernel of one of them is equal to the image of the next one and it is a non-trivial proper submodule in Θ^k_+.

Proof. (1) It is well known that Ω_c^k are irreducible and thus Θ_+^k are also irreducible. Observe that the lowest vector in Ω_c^k is $(d\xi_n)^k$ and it has the weight $(0;0,\dots,0,k)$. Now the sign changes as we go to the dual module and so we get the highest weight of Θ_c^k.

(2) By the definition of the restricted dual, it is the sum of the dual of all the graded components of the initial module. In our case Ω_c^k is the component of the minimal degree in Ω_+^k, so Θ_c^k becomes the component of the maximal degree in Θ_+^k. This implies that $L_{>0}$ acts trivially on Θ_c^k, so the morphism $\mathrm{Ind}\ \Theta_c^k \to \Theta_+^k$ is defined. Clearly Ω_+^k is isomorphic to

$$\Omega_c^k \otimes \mathbb{C}[t, \xi_1, \dots, \xi_n],$$

so it is a cofree module. Then the module Θ_+^k is a free $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_n]$-module and the morphism

$$\mathrm{Ind}\ \Theta_c^k \to \Theta_+^k$$

is therefore an isomorphism.

(3) This statement follows immediately from the fact that d commutes with the action of vector fields. □

Corollary 4.2. *The L-modules Ω_+^k of differential forms are isomorphic to the co-induced modules*

$$\Omega_+^k = \mathrm{Cnd}\ \Omega_c^k.$$

Let us now study the L-modules Ω_-^k. First, notice that these modules are free as $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_n]$-modules. Let

$$\xi_* = \xi_1 \cdots \xi_n, \qquad \text{and} \qquad \bar{\Omega}_c^k = t^{-1}\xi_*\Omega_c^k \subset \Omega_-^k. \tag{4.5}$$

Proposition 4.3. *For $L = W(1,n)_+$, we have:*

(1) $\bar{\Omega}_c^k$ is an irreducible L_0-submodule in Ω_-^k with highest weight

$$\begin{aligned} (-1;0,0,\dots,0), &\quad \textit{for } k = 0, \\ (0;k,1,\dots,1), &\quad \textit{for } k > 0, \end{aligned}$$

and $L_{>0}$ acts trivially on $\bar{\Omega}_c^k$.

(2) There is an isomorphism $\Omega_-^k = \mathrm{Ind}_{L_{\geq 0}}^L \bar{\Omega}_c^k$.

(3) The differential d gives us L-module morphisms on Ω_-^k and the kernel and image of d are L-submodules in Ω_-^k.

(4) The kernel of d and image of d in Ω_-^k for $k \geq 2$ coincide, in Ω_-^1 we have $\mathrm{Ker}\ d = \mathbb{C}(t^{-1}dt) + \mathrm{Im}\ d$, *and in Ω_-^0, we have* $\mathrm{Ker}\ d = 0$ *(and the image does not exist).*

Proof. (1) First of all, $\bar{\Omega}^k_c$ is the maximum total degree component in Ω^k_-, so any element from $L_{>0}$ moves it to zero. Also, as L_0-module $\bar{\Omega}^k_c$ is isomorphic to Ω^k_c multiplied by the 1-dimensional module $\langle t^{-1}\xi_*\rangle$. This permits us to see that its highest vectors are

$$\begin{array}{cc} \langle t^{-1}\xi_*\rangle & \text{for } k=0, \\ \langle t^{-1}\xi_* dt\rangle & \text{for } k=1, \\ \langle t^{-1}\xi_* dt(d\xi_1)^{k-1}\rangle & \text{for } k>1. \end{array}$$

The values of the highest weights are easy to compute.

(2) It is straightforward to see that Ω^0_- is a free rank 1 $\mathbb{C}[\partial_0,\partial_1,\ldots,\partial_n]$-module. Now, the action of $\partial_0,\partial_1,\ldots,\partial_n$ on Ω^k_- is coefficient wise and the fact that Ω^k_- is a free $\mathbb{C}[\partial_0,\partial_1,\ldots,\partial_n]$-module follows. This gives us the isomorphism $\Omega^k_- = \mathrm{Ind}^L_{L_{\geq 0}}\bar{\Omega}^k_c$. Parts (3) and (4) are left to the reader. $\square$

The above statement shows us that there are non-trivial submodules in $\Omega^k_\pm$ and Θ^k_+. In fact, these are "almost all" proper submodules and the respective factors are irreducible. These results are discussed in Section 4.4. In order to get this result we need to study singular vectors.

4.3 Singular vectors of W_n-modules

Having in mind the results of Section 3, we introduce the following modules. Given a $gl(1|n)$-module V, we have the associated tensor field $W(1,n)$-module $\mathbb{C}[t,t^{-1}]\otimes\Lambda(n)\otimes V$, which is a formal distribution module spanned by a collection of fields $E=\{\delta(t-z)fv \mid f\in\Lambda(n), v\in V\}$. The associated conformal W_n-module is

$$\mathrm{Tens}(V)=\mathbb{C}[\partial]\otimes(\Lambda(n)\otimes V)) \tag{4.6}$$

with the following λ-action:

$$\begin{aligned} a_\lambda(g\otimes v) = a(g)\otimes v + (-1)^{p(a)}\textstyle\sum_{i,j=1}^n(\partial_i f_j)g\otimes(E_{ij}-\delta_{ij})(v) - \\ -\lambda(-1)^{p(g)}\textstyle\sum_{j=1}^n f_j g\otimes E_{0j}(v), \end{aligned} \tag{4.7}$$

$$\begin{aligned} f_\lambda(g\otimes v) = (-\partial)(fg\otimes v) + (-1)^{p(fg)}\textstyle\sum_{i=1}^n(\partial_i f)g\otimes E_{i0}(v) + \\ +\lambda(fg\otimes E_{00}(v)). \end{aligned} \tag{4.8}$$

where $a=\sum_{i=1}^n f_i\partial_i\in W(n), f,g\in\Lambda(n), v\in V$, and $E_{ij}\in gl(1|n)$ are matrix units (they correspond to the level 0 elements $\xi_i\partial_j$ with the notation $\xi_0=t$ and $\partial_0=\partial_t$).

In this case, the modules $M(F)=\ \mathrm{Ind}^L_{L_{\geq 0}}F$ defined in Section 3, correspond to the W_n-module $\mathrm{Tens}(F)$, with F a finite-dimensional (irreducible) $gl(1|n)$-module. When we discuss the highest weight of vectors and singular vectors,

we always mean with respect to the upper Borel subalgebra in $L = W(1,n)_+$ generated by $L_{>0}$ and the elements of L_0:

$$t\partial_i, \quad \xi_i\partial_j \;\; i < j. \tag{4.9}$$

Therefore, in the module $M(V)$, viewed as a module over the annihilation algebra $W(1,n)_+$ (see Proposition 2.3), a vector $m \in M(V)$ is a singular vector if and only if the following conditions are satisfied ($g = \xi_{i_1} \cdots \xi_{i_s} \in \Lambda(n)$, and $\partial_0 = \partial_t$)

(s1) $t^n g\partial_i \cdot m = 0$ for $n > 1$,

(s2) $t^1 g\partial_i \cdot m = 0$ except for $g = 1$ and $i = 0$,

(s3) $t^0 g\partial_j \cdot m = 0$ for $s > 1$ or $g = \xi_i$ with $i < j$.

(4.10)

We shall frequently use the notation

$$\xi_I = \xi_{i_1} \cdots \xi_{i_s} \in \Lambda(n), \qquad \text{with } I = \{i_1, \ldots, i_s\}. \tag{4.11}$$

Therefore, these conditions on a singular vector $m \in$Tens(V) translate in terms of the λ-action to (cf. (2.3)):

(S1) $\frac{d^2}{d\lambda^2}(f_\lambda m) = 0$ for all $f \in \Lambda(n)$,

(S2) $\frac{d}{d\lambda}(a_\lambda m) = 0$ for all $a \in W(n)$,

(S3) $\frac{d}{d\lambda}(f_\lambda m)|_{\lambda=0} = 0$ for all $f \in \Lambda(n)$ with $f \neq 1$,

(S4) $(a_\lambda m)|_{\lambda=0} = 0$ for all $a = \xi_I\partial_j \in W(n)$ with $|I| > 1$ or $a = \xi_i\partial_j$ with $i < j$,

(S5) $(f_\lambda m)|_{\lambda=0} = 0$ for all $f = \xi_I \in \Lambda(n)$ with $|I| > 1$.

In order to classify the finite irreducible W_n-modules we should solve these equations (S1-5) to obtain the singular vectors.

Let $m \in \mathrm{Tens}(V) = \mathbb{C}[\partial] \otimes \Lambda(n) \otimes V$, then

$$m = \sum_{k=0}^{N} \sum_I \partial^k(\xi_I \otimes v_{I,k}), \qquad \text{with } v_{I,k} \in V. \tag{4.12}$$

In order to obtain the singular vectors, we need the some reduction lemmas:

Lemma 4.4. *If $m \in \mathrm{Tens}(V)$ is a singular vector, then the degree of m in ∂ is at most 1.*

Proof. Using (4.7) , we have for $a = \sum_{i=1}^{n} f_i \partial_i$ that

$$(a_\lambda m)' = \sum_{k=1}^{N} \sum_I k(\lambda + \partial)^{k-1} \Bigg[a(\xi_I) \otimes v_{I,k} \tag{4.13}$$

$$+(-1)^{p(a)} \sum_{i,j=1}^{n} (\partial_i f_j) \xi_I \otimes (E_{ij} - \delta_{ij})(v_{I,k}) - \lambda(-1)^{|I|} \sum_{j=1}^{n} f_j \xi_I \otimes E_{0j}(v_{I,k}) \Bigg]$$

$$- \sum_{k=0}^{N} \sum_I (\lambda + \partial)^k (-1)^{|I|} \sum_{j=1}^{N} f_j \xi_I \otimes E_{0j}(v_{I,k}).$$

Taking $a = \partial_j$, condition (S2) become

$$\begin{aligned} 0 = \sum_{k=1}^{N} \sum_{I | j \in I} k(\lambda + \partial)^{k-1} (\xi_{i_1} \cdots \hat{\xi}_j \cdots \xi_{i_s} \otimes v_{I,k}) \\ -\lambda \sum_{k=1}^{n} \sum_I (-1)^{|I|} k(\lambda + \partial)^{k-1} (\xi_I \otimes E_{0j}(v_{I,k})) \\ - \sum_{k=0}^{N} \sum_I (\lambda + \partial)^k (-1)^{|I|} (\xi_I \otimes E_{0j}(v_{I,k})). \end{aligned} \tag{4.14}$$

Now, viewed as a polynomial in λ, we obtain

$$E_{0j}(v_{I,k}) = 0, \qquad \forall I, k = 1, \ldots, N, \text{ and } j = 1, \ldots, n. \tag{4.15}$$

Using it in (4.14) and taking the coefficients in $\lambda + \partial$, we get

$$v_{I,k} = 0 \qquad \text{for all } I \neq \emptyset, \text{ and } k \geq 2.$$

Hence, $m = \sum_{k=0}^{1} \sum_I \partial^k (\xi_I \otimes v_{I,k}) + \sum_{k=2}^{N} \partial^k (1 \otimes v_{\emptyset,k})$.

Using (4.8) and condition (S1) for $f = 1$, we have

$$\begin{aligned} 0 = (f_\lambda m)\text{''} = 2 \sum_I (\xi_I \otimes E_{00}(v_{I,1})) \\ - \sum_{k=2}^{N} (k-1) k (\lambda + \partial)^{k-2} \partial (1 \otimes v_{\emptyset,k}) \\ + \sum_{k=2}^{N} \left(2k(\lambda + \partial)^{k-1} + \lambda k(k-1)(\lambda + \partial)^{k-2} \right) (1 \otimes E_{00}(v_{\emptyset,k})) \end{aligned} \tag{4.16}$$

Then, viewed as a polynomial in λ, we have $E_{00}(v_{\emptyset,k})$ for all $k \geq 2$. Hence the last term in (4.16) is 0. Now, viewed as a polynomial in $(\lambda + \partial)$, we obtain $v_{\emptyset,k} = 0$ for all $k \geq 2$, finishing the proof. $\square$

Observe that the coefficient in $(\lambda+\partial)^0$ in (4.16), gives us the following useful identity

$$E_{00}(v_{I,1}) = 0 \qquad \text{for all } I. \tag{4.17}$$

We will use the following notation: $[1, n] = \{1, \ldots, n\}$.

Lemma 4.5. *If m is a singular vector, then*

$$m = \partial(\xi_{[1,n]} \otimes w) + \sum_{l=1}^{n} (\xi_{[1,n]-\{l\}} \otimes v_l) + \xi_{[1,n]} \otimes v_0.$$

Proof. By the previous Lemma, we have

$$m = \sum_I \Big[\partial(\xi_I \otimes v_{I,1}) + (\xi_I \otimes v_{I,0}) \Big]$$

Now (S5) gives us

$$\begin{aligned} 0 = (f_\lambda m)_{|_{\lambda=0}} = \\ = \textstyle\sum_I \Big((-\partial)(f\xi_I \otimes v_{I,0}) - (-1)^{|I|} \sum_{i=1}^n (\partial_i f)\xi_I \otimes E_{i0}(v_{I,0}) \Big) \\ + \textstyle\sum_I \Big((-\partial^2)(f\xi_I \otimes v_{I,1}) - (-1)^{|I|} \sum_{i=1}^n \partial((\partial_i f)\xi_I \otimes E_{i0}(v_{I,1})) \Big) \end{aligned} \tag{4.18}$$

for any $f = \xi_J$ with $|J| > 1$. Considering the coefficient of ∂^2 and taking $f = \xi_l \xi_k$, we obtain $v_{I,1} = 0$ for all I with $|I| \leq n-2$. Using this and considering the coefficient of ∂ with $f = \xi_l \xi_k \xi_s$, we obtain $v_{I,0} = 0$ for all I with $|I| \leq n-3$. With this reduction, the coefficient of ∂ with $f = \xi_i \xi_j$ $(i \neq j)$ is

$$0 = -(\xi_{[1,n]} \otimes v_{[1,n]-\{i,j\},0}) + (-1)^{n-1}(\xi_{[1,n]} \otimes E_{i0}(v_{[1,n]-\{j\},1})),$$

obtaining

$$E_{i0}(v_{[1,n]-\{j\},1}) = (-1)^{n-1} v_{[1,n]-\{i,j\},0} \qquad \text{for all } i \neq j. \tag{4.19}$$

Computing (S3), we have

$$\begin{aligned} 0 = (f_\lambda m)'_{|_{\lambda=0}} = \\ = \textstyle\sum_{|I|\geq n-1} \Big((-\partial)(f\xi_I \otimes v_{I,1}) - (-1)^{|I|} \sum_{i=1}^n (\partial_i f)\xi_I \otimes E_{i0}(v_{I,1}) \Big) \\ + \partial \ \textstyle\sum_{|I|\geq n-1} f\xi_I \otimes E_{00}(v_{I,1}) + \sum_{|I|\geq n-2} f\xi_I \otimes E_{00}(v_{I,0}). \end{aligned}$$

Now, taking $f = \xi_i$, using (4.17) and considering the coefficient in ∂, we have

$$v_{I,1} = 0 \qquad \text{for all } |I| = n-1,$$

and using it in (4.19), we have

$$v_{I,0} = 0 \qquad \text{for all } |I| = n-2,$$

finishing the proof. □

Let $\xi_* := \xi_{[1,n]}$ and $\xi^l := \xi_{[1,n]-\{l\}}$. Due to the previous lemma, any singular vector has the form

$$m = \partial(\xi_* \otimes w) + \sum_{l=1}^n (\xi^l \otimes v_l) + \xi_* \otimes v_0.$$

Then, it is easy to see that conditions (s1-3) are equivalent to the following list

(s1):

$$E_{00}(w) = 0, \tag{4.20}$$
$$E_{0i}(w) = 0, \qquad i = 1, \dots, n. \tag{4.21}$$

(s2):

$$E_{ji}(w) + E_{0i}(v_j) = 0, \qquad i, j = 1, \dots, n, \tag{4.22}$$
$$E_{0i}(v_0) = 0, \qquad i = 1, \dots, n, \tag{4.23}$$
$$E_{0i}(v_j) = 0, \qquad i, j = 1, \dots, n; \quad , i \neq j, \tag{4.24}$$
$$E_{0j}(v_j) = -w, \qquad j = 1, \dots, n. \tag{4.25}$$
$$E_{i0}(w) = E_{00}(v_i), \qquad i = 1, \dots, n, \tag{4.26}$$

(s3):

$$E_{i0}(v_j) = E_{j0}(v_i), \qquad i, j = 1, \dots, n; \quad i \neq j. \tag{4.27}$$
$$E_{ij}(v_l) = E_{lj}(v_i), \qquad i, j, l = 1, \dots, n; \quad i \neq l. \tag{4.28}$$
$$E_{ij}(w) = 0, \qquad i, j = 1, \dots, n; \quad i < j, \tag{4.29}$$
$$E_{ij}(v_0) = 0, \qquad i, j = 1, \dots, n; \quad i < j, \tag{4.30}$$
$$E_{ij}(v_l) = 0, \qquad i, j, l = 1, \dots, n; \quad i < j, l \neq j, \tag{4.31}$$
$$E_{ij}(v_j) = v_i, \qquad i, j = 1, \dots, n; \quad i < j. \tag{4.32}$$

Now replacing (4.24) and (4.25) on (4.22), we obtain

$$E_{ij}(w) = \delta_{ij} \ w, \qquad i, j = 1, \dots, n. \tag{4.33}$$

Recall that we are considering the basis ($\partial_0 = \partial_t$)

$$t\partial_0; t\partial_0 + \xi_1\partial_1, \dots, t\partial_0 + \xi_n\partial_n$$

for the Cartan subalgebra H in $W(1,n)_+$, and we write the weight of an eigenvector for the Cartan subalgebra H as a tuple

$$\bar{\mu} = (\mu; \lambda_1, \dots, \lambda_n) \tag{4.34}$$

for the corresponding eigenvalues of the basis.

Using the above conditions, we can prove the following

Proposition 4.6. *Let $n \geq 2$ and m be a non-trivial singular vector in* Tens V *with weight $\bar{\mu}_m$, then we have one of the following:*

(a) $m = \xi^n \otimes v_n$, $\bar{\mu}_m = (0; 0, \dots, 0, -k)$ with $k \geq 0$, v_n is a highest weight vector in V with weight $(0; 0, \dots, 0, -k-1)$, and m is uniquely defined by v_n.

(b) $m = \sum_{l=1}^n \xi^l \otimes v_l$, $\bar{\mu}_m = (0; k, 1, \dots, 1)$ with $k \geq 2$, v_1 is a highest weight vector in V with weight $(0; k-1, 1, \dots, 1)$, and m is uniquely defined by v_1.

(c) $m = \partial(\xi_* \otimes w) + \sum_{l=1}^{n} \xi^l \otimes v_l$, $\bar{\mu}_m = (-1; 0, \ldots, 0)$, *$w$ is a highest weight vector in V with weight $(0; 1, \ldots, 1)$, and m is uniquely defined by w.*

Proof. By computing $E_{00} \cdot m = (t\partial) \cdot m$ and using (4.20) and (4.26) on it, we obtain the following conditions:
If $w = 0$, then

$$E_{00} \cdot m = 0 \quad \text{and} \quad E_{00}(v_0) = 0. \tag{4.35}$$

If $w \neq 0$, then

$$E_{00} \cdot m = -m \tag{4.36}$$
$$E_{00}(v_l) = -v_l, \qquad l = 0, \ldots, n, \tag{4.37}$$

and using (4.26), in this case ($w \neq 0$) we have

$$E_{i0}(w) = -v_i, \qquad i = 1, \ldots, n. \tag{4.38}$$

Similarly, observe that $E_{ii} \cdot m = (\xi_i \partial_i) \cdot m$. Now this action can be easily computed, and using (4.33) on it, we have the following:

If $w \neq 0$, then

$$\begin{aligned} E_{ii} \cdot m &= m, & i &= 1, \ldots, n, \\ E_{ii}(v_l) &= v_l, & l, i &= 1, \ldots, n, l \neq i, \\ E_{ii}(v_i) &= 2v_i, & i &= 1, \ldots, n. \\ E_{ii}(v_0) &= v_0, & i &= 1, \ldots, n. \end{aligned} \tag{4.39}$$

Using this and equations (4.36), (4.20) and (4.33), we obtain for the case $w \neq 0$, that the corresponding weights are

$$\bar{\mu}_m = (-1; 0, \ldots, 0) \qquad \text{and} \qquad \bar{\mu}_w = (0; 1, \ldots, 1).$$

This result together with (4.38) give us the proof of case (c) in the proposition.

For the rest of the proof, we assume $w = 0$, let us show that the only possible cases are (a) and (b).

Observe that replacing (4.32) in (4.28), we get

$$E_{jj}(v_i) = v_i \qquad \forall i < j. \tag{4.40}$$

Now, equation (4.32) shows that if $v_l = 0$ for some l with $1 \leq l \leq n$, then $v_j = 0$ for all $j < l$. In order to finish the proof, we should show that only the two extreme cases are possible, that is $v_l \neq 0$ for all l or $v_l = 0$ except for $l = n$.

Now, suppose that there exist $l > 1$ such that $v_j = 0$ for all $j < l$ and $v_l \neq 0$, then using (4.28) we have that

$$E_{ii}(v_k) = 0, \qquad i < l \leq k. \tag{4.41}$$

Then by (4.41) and (4.40), we obtain that

$$E_{ii} \cdot m = \alpha\, m \quad \text{with } \alpha = 0 \text{ or } 1, \qquad \text{if } i < l \text{ or } i > l, \text{ respectively.}$$

Therefore, using this and (4.35) we get

$$\bar{\mu}_m = (0; 0, \ldots, 0, k, 1, \ldots, 1)$$

where $E_{ll} \cdot m = k\ m$. But the space V, from which we are inducing is finite-dimensional and a singular vector generates a finite-dimensional L_0-submodule, then (recall notation (4.34))

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

is a highest weight, and because of that only two extreme positions of k are possible (recall that $n > 1$). This gives us the cases (a) and (b). In order to finish the proof we need to complete the computation of weights in each case.

If $v_1 \neq 0$, then using (4.26) and (4.40) we obtain

$$\bar{\mu}_m = (0; k, 1, \ldots, 1) \qquad \text{and} \qquad \bar{\mu}_{v_1} = (0; k-1, 1, \ldots, 1), \qquad \text{with } k \geq 1.$$

getting case (b).

If $v_l = 0$ except for $l = n$, then using (4.26) and (4.41) we obtain

$$\bar{\mu}_m = (0; 0, \ldots, 0, k) \qquad \text{and} \qquad \bar{\mu}_{v_n} = (0; 0, \ldots, 0, k-1), \qquad \text{with } k \leq 0.$$

obtaining case (c). Case (d) is immediate. □

4.4 Irreducible induced $W(1, n)_+$-modules

In this subsection we consider $L = W(1, n)_+$, with $n \geq 2$. Now, we have the following:

Theorem 4.7. *Let $n \geq 2$ and F be an irreducible L_0-module with highest weight $\bar{\mu}_*$. Then the L-modules $\mathrm{Ind}_{L_{\geq 0}}^{L} F$ are irreducible finite continuous modules except for the following cases:*

(a) $\bar{\mu}_* = (0; 0, \ldots, 0, -m), m \geq 0$, where $\mathrm{Ind}_{L_{\geq 0}}^{L} F = \Theta_+^m$ and the image $d^{\#}\Theta_+^{m+1}$ is the only non-trivial proper submodule.

(b) $\bar{\mu}_* = (0; k, 1, \ldots, 1), k \geq 1$, where $\mathrm{Ind}_{L_{\geq 0}}^{L} F = \Omega_-^k$. For $k \geq 2$ the image $d\Omega_-^{k-1}$ is the only non-trivial proper submodule. For $k = 1$, both $\mathrm{Im}(d)$ *and* $\mathrm{Ker}(d)$ *are proper submodules.* $\mathrm{Ker}(d)$ *is a maximal submodule.*

Remark 4.8. Let F be an irreducible L_0-module with highest weight $\bar{\mu}_* = (-1; 0, \ldots, 0)$. Then $\mathrm{Ind}_{L_{\geq 0}}^{L} F = \Omega_-^0$ is an irreducible L-module. Note that the image of $d : \Omega_-^0 \to \Omega_-^1$ is the submodule in Ω_-^1 generated by the singular vector correponding to the case (c) in Proposition 4.6, but it is not a maximal submodule (see Proposition 4.3 (4)).

Proof. We know from Theorem 3.1 that in order for $\mathrm{Ind}_{L_{\geq 0}}^{L} F$ to be reducible it has to have non-trivial singular vectors and the possible highest weights of F in this situation are listed in Proposition 4.6 above.

The fact that the induced modules are actually reducible in those cases is known because we have got nice realizations for these induced modules in Propositions 4.1 and 4.3 together with morphisms defined by $d, d^{\#}$, so kernels and images of these morphisms become submodules.

The subtle thing is to prove that a submodule is really a maximal one. We notice that in each case the factor is isomorphic to a submodule in another induced module so it is enough to show that the submodule is irreducible. This can be proved as follows, a submodule in the induced module is irreducible if it is generated by any highest singular vector that it contains. We see from our list of non-trivial singular vectors that there is at most one such a vector for each case and the images and kernels in question are exactly generated by those vectors, hence they are irreducible. □

Corollary 4.9. *The theorem gives us a description of finite continuous irreducible $W(1,n)_+$-modules for $n \geq 2$. Such a module is either $\mathrm{Ind}_{L_{\geq 0}}^{L} F$ for an irreducible finite-dimensional L_0-module F where the highest weight of F does not belong to the types listed in (a), (b) of the theorem or the factor of an induced module from (a), (b) by its submodule $Ker(d)$.*

4.5 Finite irreducible W_n-modules

In order to give an explicit construction and classification, we need the following notation. Recall that $W(1,n)$ acts by derivations on the algebra of differential forms $\Omega = \Omega(1,n)$, and note that this is a conformal module by taking the family of formal distributions

$$E = \{\delta(z-t)\omega \text{ and } \delta(z-t)\omega \; dt \mid \omega \in \Omega(n)\}$$

Translating this and all other attributes of differential forms, like de Rham differential, etc. into the conformal algebra language, we arrive to the following definitions.

Recall that given an algebra A, the associated current formal distribution algebra is $A[t,t^{-1}]$ with the local family $F = \{a(z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1} = a\delta(z-t)\}_{a\in A}$. The associated conformal algebra is $\mathrm{Cur}A = \mathbb{C}[\partial] \otimes A$ with multiplication defined by $a_\lambda b = ab$ for $a, b \in A$ and extended using sesquilinearity. This is called the *current conformal algebra*, see [7] for details.

The conformal algebra of differential forms Ω_n is the current algebra over the commutative associative superalgebra $\Omega(n) + \Omega(n)\; dt$ with the obvious multiplication and parity, subject to the relation $(dt)^2 = 0$:

$$\Omega_n = \mathrm{Cur}(\Omega(n) + \Omega(n)\; dt).$$

The de Rham differential $\widetilde{d}$ of Ω_n (we use the tilde in order to distinguish it from the de Rham differential d on $\Omega(n)$) is a derivation of the conformal algebra Ω_n

such that:

$$\widetilde{d}(\omega_1 + \omega_2 dt) = d\omega_1 + d\omega_2 dt - (-1)^{p(\omega_1)}\partial(\omega_1 dt). \tag{4.42}$$

here and further $\omega_i \in \Omega(n)$.

The standard $\mathbb{Z}_+$-gradation $\Omega(n) = \oplus_{j\in\mathbb{Z}_+}\Omega(n)^j$ of the superalgebra of differential forms by their degree induces a $\mathbb{Z}_+$-gradation

$$\Omega_n = \oplus_{j\in\mathbb{Z}_+}\Omega_n^j, \qquad \text{where } \Omega_n^j = \mathbb{C}[\partial]\otimes(\Omega(n)^j + \Omega(n)^{j-1}\ dt),$$

so that $\widetilde{d} : \Omega_n^j \to \Omega_n^{j+1}$.

The contraction ι_D for $D = a + f \in W_n$ is a conformal derivation of Ω_n such that:

$$\begin{aligned} (\widetilde{L}_a)_\lambda(\omega_1 + \omega_2 dt) &= L_a\omega_1 + (L_a\omega_2)dt, \\ (\widetilde{L}_f)_\lambda\omega &= -(\partial+\lambda)(f\omega), \\ (\widetilde{L}_f)_\lambda(\omega dt) &= (-1)^{p(f)+p(\omega)}(df)\omega - \partial(f\omega dt). \end{aligned} \tag{4.43}$$

The properties of $\Omega(1,n)$ imply the corresponding properties of Ω_n given by the following proposition.

Proposition 4.10. *(a)* $\widetilde{d}^2 = 0$.

(b) The complex $(\Omega_n, \widetilde{d}) = \{0 \to \Omega_n^0 \to \cdots \to \Omega_n^j \to \cdots\}$ *is exact at all terms* Ω_n^j*, except for* $j = 1$*. One has:* $\mathrm{Ker}\ \widetilde{d}_{|\Omega_n^1} = \widetilde{d}\Omega_n^0 \oplus \mathbb{C}dt$.

(c) $\iota_{D_1}\iota_{D_2} + p(D_1, D_2)\iota_{D_2}\iota_{D_1} = 0$.

(d) $\widetilde{L}_D\widetilde{d} = (-1)^{p(D)}\widetilde{d}\widetilde{L}_D$.

(e) $\widetilde{L}_D = \widetilde{d}\iota_D + (-1)^{p(D)}\iota_D\widetilde{d}$.

(f) The map $D \mapsto \widetilde{L}_D$ *defines a* W_n*-module structures on* Ω_n*, preserving the* $\mathbb{Z}_+$*-gradation and commuting with* $\widetilde{d}$.

Proof. Only the proof of (b) requires a comment. Following Proposition 3.2.2 of [7], we construct $\mathbb{C}[\partial]$-linear maps $K : \Omega_n \to \Omega_n$ (a homotopy operator) and $\epsilon : \Omega_n \to \Omega_n$ by the formulas ($\omega \in \Omega(n) + \Omega(n)dt$):

$$\begin{aligned} K(d\xi_n\omega) = \xi_n\omega, \quad K(\omega) = 0 \qquad &\text{if } \omega \text{ does not involve } d\xi_n, \\ \epsilon(d\xi_n\omega) = \epsilon(\xi_n\omega) = 0, \quad \epsilon(\omega) = \omega \qquad &\text{if } \omega \text{ does not involve both } d\xi_n \text{ and } \xi_n. \end{aligned}$$

One checks directly that

$$K\widetilde{d} + \widetilde{d}K = 1 - \epsilon.$$

Therefore, if $\omega \in \Omega_n$ is a closed form, we get $\omega = \widetilde{d}(K\omega) + \epsilon(\omega)$. It follows by induction on n that $\omega = \widetilde{d}\omega_1 + P(\partial)dt$ for some $\omega_1 \in \Omega_n$ and a polynomial $P(\partial)$. But it is clear from (4.42) that $P(\partial)dt$ is always closed, and it is exact iff $P(\partial)$ is divisible by ∂. □

Since the extended annihilation algebra $W(1,n)^+$ is a direct sum of $W(1,n)_+$ and a 1-dimensional Lie algebra $\mathbb{C}a$, any irreducible $W(1,n)^+$-module is obtained from a $W(1,n)_+$-module M by extending to $W(1,n)^+$, letting $a \mapsto -\alpha$, where $\alpha \in \mathbb{C}$. Translating into the conformal language (see Proposition 2.3), we see that all W_n-modules are obtained from conformal $W(1,n)_+$-modules by taking for the action of ∂ the action of $-\partial_t + \alpha I, \alpha \in \mathbb{C}$. We denote by $\mathrm{Tens}_\alpha V$ and $\Omega_{k,\alpha}, \alpha \in \mathbb{C}$, the W_n-modules obtained from $\mathrm{Tens}V$ and Ω_k by replacing in (4.7) and (4.8) respectively ∂ by $\partial + \alpha$.

Now, Theorem 4.7 and Corollary 4.9, along with Section 3 and Propositions 2.3, 2.8, 2.6 and 2.9 give us a complete description of finite irreducible W_n-modules.

Theorem 4.11. *The following is a complete list of non-trivial finite irreducible W_n-modules ($n \geq 2, \alpha \in \mathbb{C}$):*

(a) $\mathrm{Tens}_\alpha V$*, where V is a finite-dimensional irreducible $gl(1|n)$-module different from $\Lambda^k(\mathbb{C}^{1|n})^*$, $k = 1, 2, \dots$ and $\bar{\Omega}_c^k$ (see (4.5)), $k = 1, 2, ...,$*

(b) $\Omega^*_{k,\alpha}/\mathrm{Ker}\ \widetilde{d}^*, k = 1, 2, \dots$ *, and the same modules with reversed parity,*

(c) W_n-modules dual to (b)*, with $k > 1$.*

Remark 4.12. (a) Using Proposition 4.3, we have that the kernel of $\widetilde{d}$ and the image of $\widetilde{d}$ coincide in Ω_k for $k \geq 2$. Now, since Ω_{k+2} is a free $\mathbb{C}[\partial]$-module of finite rank and $\Omega_{k+1}/\mathrm{Im}\widetilde{d} = \Omega_{k+1}/\mathrm{Ker}\widetilde{d} \simeq \mathrm{Im}\widetilde{d} \subset \Omega_{k+2}$, we obtain that $\Omega_{k+1}/\mathrm{Im}\ \widetilde{d}$ is a finitely generated free $\mathbb{C}[\partial]$-module. Therefore, we can apply Proposition 2.6, and we have that

$$\Omega^*_{k+1,\alpha}/\mathrm{Ker}\ \widetilde{d}^* \simeq \left(\Omega_{k,\alpha}/\mathrm{Ker}\ \widetilde{d}\right)^* \tag{4.44}$$

for $k \geq 1$.

(b) Observe that we can not apply the previous argument for $k = 0$ since, by Proposition 4.3, the image of $\widetilde{d}$ has codimension one (over $\mathbb{C}$) in Ker $\widetilde{d}$. In fact, (4.44) is not true for $k = 0$. For example, this can be easily seen for $W_0 = Vir$ using the differential map which is explicitly written in Remark 2.5.

(c) Observe that $\Omega_{0,\alpha}$ is an irreducible tensor module (Ker $\widetilde{d} = 0$, cf. Proposition 4.3), that is why this module is included in case (a) of Theorem 4.11.

(d) Since for a finite rank module M over a Lie conformal superalgebra we have $M^{**} = M$ (see Proposition 2.7), the W_n-modules in case (c) of Theorem 4.11 are isomorphic to $\Omega_{k,\alpha}/ker\ \widetilde{d}$, $k = 2, 3, ...$

(e) Observe that $(\mathrm{TensV})^*$ is not isomorphic to TensV^*. For example, consider the case of W_1. We have, using the notation below, that $M(a,b) =$Tens $V_{a,b}$. It is easy to see that for the case $a + b \neq 0$, $(\mathrm{TensV_{a,b}})^* =$Tens $V_{-a,-b}$, but $(V_{a,b})^* = V_{1-a,-b-1}$.

Now we will present the case $n = 1$ in detail and we shall see that our result agrees with the classification given in [4] for $K_2 \simeq W_1$. Let us fix some notations. We have

$$W_1 = \mathbb{C}[\partial] \otimes (\Lambda(1) \oplus W(1)) = \mathbb{C}[\partial]\{1, \xi, \partial_1, \xi\partial_1\}.$$

In [4], the conformal Lie superalgebra K_2 is presented as the freely generated module over $\mathbb{C}[\partial]$ by $\{L, J, G^{\pm}\}$. An isomorphism between K_2 and W_1 is explicitly given by

$$L \mapsto -1 + \frac{1}{2}\partial\xi\partial_1, \qquad J \mapsto \xi\partial_1, \qquad G^+ \mapsto 2\xi, \qquad G^- \mapsto -\partial_1. \tag{4.45}$$

The irreducible modules of W_1 are parameterized by finite-dimensional irreducible representations of $gl(1,1)$ (and the additional twist by alpha that, for simplicity, shall be omitted in the formulas below). The irreducible representations of $gl(1,1)$, denoted by $V_{a,b}$, are parameterized by a and b, the corresponding eigenvalues of e_{11} and e_{22} on the highest weight vector.

If both parameters are equal to zero, the representation is trivial 1-dimensional. Otherwise, either $a + b = 0$, the dimension of the $gl(1,1)$-representation is 1, and the corresponding representation of W_1 is one of the tensor modules of rank 2. Or else $a + b$ is non-zero, the dimension of the $gl(1,1)$-representation is 2, and the corresponding tensor module has rank 4.

Explicitly, consider the set of $\mathbb{C}[\partial]$-generators of W_1 $\{1, \xi, \partial_1, \xi\partial_1\}$. Let a and b such that $a + b \neq 0$. Let $V_{a,b} = \mathbb{C}\text{-span}\{v_0, v_1\}$, where v_0 is a highest weight vector. Let $M(a,b) = M(V_{a,b}) = \mathbb{C}[\partial]\{v_0, v_1, w_1 = \partial_1 v_0, w_0 = \partial_1 v_1\}$ be the tensor W_1-module and denote by $L(a,b)$ the irreducible quotient. The action of W_1 in $M(a,b)$ is given explicitly by the following formulas:

$$\begin{aligned} 1_\lambda v_0 &= (a\lambda - \partial)v_0, & 1_\lambda v_1 &= ((a-1)\lambda - \partial)v_1, \\ 1_\lambda w_1 &= (a\lambda - \partial)w_1, & 1_\lambda w_0 &= ((a-1)\lambda - \partial)w_0, \end{aligned}$$

$$\begin{aligned} \xi_\lambda v_0 &= v_1, & \xi_\lambda v_1 &= 0, \\ \xi_\lambda w_1 &= (a\lambda - \partial)v_0 - w_0, & \xi_\lambda w_0 &= ((a-1)\lambda - \partial)v_1, \end{aligned}$$

$$\begin{aligned} \partial_{1\lambda} v_0 &= w_1, & \partial_{1\lambda} v_1 &= (a+b)\lambda v_0 + w_0, \\ \partial_{1\lambda} w_1 &= 0, & \partial_{1\lambda} w_0 &= -(a+b)\lambda w_1 \end{aligned}$$

$$\begin{aligned} \xi\partial_{1\lambda} v_0 &= b\, v_0, & \xi\partial_{1\lambda} v_1 &= (b+1)\, v_1, \\ \xi\partial_{1\lambda} w_1 &= (b-1) w_1, & \xi\partial_{1\lambda} w_0 &= -(a+b)\lambda\, v_0 + b\, w_0. \end{aligned} \tag{4.46}$$

If $a + b \neq 0$ and $a \neq 0$, then $M(a,b)$ is irreducible of rank 4, and the explicit action is given by (4.45). The proof of this statement is in the following way (the proof of the other statements below are much simpler): Take $v = p(\partial)v_0 + q(\partial)w_0 + r(\partial)v_1 + s(\partial)w_1$ in a submodule of $M(a,b)$. Denote by w

the coefficient of the highest power in λ of $\xi_\lambda v$ and by y the coefficient of the highest power in λ of $\xi_\lambda w$.

If $a \neq 1$ then $y = v_1$ (up to a constant factor), therefore v_1 lies in the submodule. If $a = 1$, then by taking the coefficient of the highest power in λ of $\xi\partial_{1\lambda} y$ and using that in this case $b \neq -1$, we also obtain that v_1 lies in the submodule.

Therefore, in any case we have that v_1 lies in any submodule, and by the formulas for the actions on v_1 it is immediate that the other generators also belong to any submodule, proving that $M(a,b)$ is irreducible in this case.

If $a + b \neq 0$ but $a = 0$, it is easy to show as above that $N = \mathbb{C}[\partial]w_1 \oplus \mathbb{C}[\partial](\partial v_0 + w_0)$ is a submodule of $M(0,b)$. The irreducible quotient of $M(0,b)$ by N is $L(0,b) = \mathbb{C}[\partial]v_0 \oplus \mathbb{C}[\partial]v_1$, of rank 2, and the action here is explicitly, as follows:

$$\begin{aligned} 1_\lambda v_0 &= (-\partial)v_0, & 1_\lambda v_1 &= (-\lambda - \partial)v_1, \\ \xi_\lambda v_0 &= v_1, & \xi_\lambda v_1 &= 0, \\ \partial_{1\lambda} v_0 &= 0, & \partial_{1\lambda} v_1 &= (b\lambda - \partial)v_0, \\ \xi\partial_{1\lambda} v_0 &= b\, v_0, & \xi\partial_{1\lambda} v_1 &= (b+1)\, v_1. \end{aligned} \tag{4.47}$$

If $a + b = 0$, but $a \neq 0$, it is easy to show as above that $M(a,-a) = C[\partial]\{v_0, w_1\}$ is irreducible of rank 2 and the action of W_1 here is given by:

$$\begin{aligned} 1_\lambda v_0 &= (a\lambda - \partial)v_0, & 1_\lambda w_1 &= (a\lambda - \partial)w_1, \\ \xi_\lambda v_0 &= 0, & \xi_\lambda w_1 &= (a\lambda - \partial)v_0, \\ \partial_{1\lambda} v_0 &= w_1, & \partial_{1\lambda} w_1 &= 0, \\ \xi\partial_{1\lambda} v_0 &= -a\, v_0, & \xi\partial_{1\lambda} w_1 &= (-a-1)w_1. \end{aligned} \tag{4.48}$$

Thus we obtain

Corollary 4.13. *The W_1-module $L(a,b)$ as a $\mathbb{C}[\partial]$-module has rank: 4 if $a+b \neq 0$ and $a \neq 0$, 2 if $a+b \neq 0$ and $a = 0$, 2 if $a+b = 0$ and $a \neq 0$, 0 if $a = b = 0$. These are all non-trivial finite irreducible W_1-modules.*

Remark 4.14. In [4], the irreducible representations of K_2 are classified in terms of parameters Λ and Δ. Using the isomorphism between K_2 and W_1 in (4.44), these parameters are related to ours as follows,

$$a = -\Delta - \frac{\Lambda}{2}\,,\ b = \Lambda.$$

Then it can be easily checked that the above corollary corresponds to Theorem 4.1 in [4], and explicit formulas for the λ- action given at the end of section 4 in [4], corresponds to ours in each case.

5 Lie conformal superalgebra S_n and its finite irreducible modules

Recall that the *divergence* of a differential operator $D = \sum_{i=0}^{n} a_i \partial_i \in W(1,n)$, with $a_i \in \Lambda(1,n)$ and $\partial_0 = \partial_t$ is defined by the formula

$$div\ D = \partial_0 a_0 + \sum_{i=1}^{n} (-1)^{p(a_i)} \partial_i a_i.$$

The basic property of the divergence is $(D_1, D_2 \in W(1,n))$

$$div\ [D_1, D_2] = D_1(div\ D_2) - (-1)^{p(D_1)p(D_2)} D_2(div\ D_1).$$

It follows that

$$S(1,n) = \{D \in W(1,n)\ :\ div\ D = 0\}$$

is a subalgebra of the Lie superalgebra $W(1,n)$. Similarly,

$$S(1,n)_+ = \{D \in W(1,n)_+\ :\ div\ D = 0\}$$

is a subalgebra of $W(1,n)_+$. We have

$$S(1,n)\ (\text{resp. } S(1,n)_+\ ) = S(1,n)'\ (\text{resp. } S(1,n)'_+\ ) \oplus \mathbb{C}\xi_1 \cdots \xi_n \partial_0, \quad (5.1)$$

where $S(1,n)'$ (resp. $S(1,n)'_+$) denotes the derived subalgebra. It is easy to see that $S(1,n)'$ is a formal distribution Lie superalgebra, see [6], Example 3.5.

In order to describe the associated Lie conformal superalgebra, we need to translate the notion of divergence to the "conformal" language as follows. It is a $\mathbb{C}[\partial]$-module map $div : W_n \to \text{Cur}\ \Lambda(n)$, given by

$$div\ a = \sum_{i=1} (-1)^{p(f_i)} \partial_i f_i, \qquad div\ f = -\partial \otimes f,$$

where $a = \sum_{i=1}^{n} f_i \partial_i \in W(n)$ and $f \in \Lambda(n)$. The following identity holds in $\mathbb{C}[\partial] \otimes \Lambda(n)$, where $D_1, D_2 \in W_n$:

$$div\ [{D_1}_\lambda D_2] = (D_1)_\lambda (div\ D_2) - (-1)^{p(D_1)p(D_2)} (D_2)_{-\lambda-\partial} (div\ D_1). \quad (5.2)$$

Therefore,

$$S_n = \{D \in W_n\ :\ div\ D = 0\}$$

is a subalgebra of the Lie conformal superalgebra W_n. It is known that S_n is simple for $n \geq 2$, and finite of rank $n2^n$. Furthermore, it is the Lie conformal superalgebra associated to the formal distribution Lie superalgebra $S(1,n)'$. The annihilation algebra and the extended annihilation algebra is given by

$$\mathcal{A}(S_n) = S(1,n)'_+ \qquad \text{and} \qquad \mathcal{A}(S_n)^e = \mathbb{C}\ ad(\partial_0) \ltimes S(1,n)'_+.$$

Now, we have to study representations of $S(1,n)_+$ and of its derived algebra $S(1,n)'_+$ which has codimension 1. Observe that $S(1,n)_+$ inherits the

$\mathbb{Z}$-gradation in $W(1,n)_+$, and denoting by $L = S(1,n)_+$ (for the rest of this section), we have that $L_{-1} =< \partial_0, \dots, \partial_n >$ as in $W(1,n)_+$ but the other graded components are strictly smaller than these of $W(1,n)_+$. Observe that $L_0 = sl(1|n)$.

In order to consider weights of vectors in $S(1,n)_+$-modules, we take the basis

$$t\partial_0 + \xi_1\partial_1, \dots, t\partial_0 + \xi_n\partial_n.$$

for the Cartan subalgebra. And the weights are written as $\bar{\lambda} = (\lambda_1, \dots, \lambda_n)$ for the corresponding eigenvalues.

Propositions 4.1 and 4.3, and Corollary 4.2 holds for $L = S(1,n)_+$ with the following minor modification: all highest weights are the same as in the W case, except for the first coordinate that should be removed.

Similarly, if V is an $sl(1|n)$-module, then formulas (4.7) and (4.8) define an S_n-module structure in $\mathrm{Tens}(V)$. Indeed, elements $(-1)^{p(f)}\partial(f\partial_i) + \partial_i f$, with $f \in \Lambda(n)$ generate S_n as a $\mathbb{C}[\partial]$-module, and it is easy to see that for the action of these elements defined by (4.7) and (4.8), one needs only $E_{ij}(v)$ for $i \neq j$ and $(E_{00} + E_{ii})(v)$ for $i > 0$.

As in the W-case, the classification is reduced to the study of singular vectors in $\mathrm{Tens}(V)$, where V is an $sl(1|n)$-module. Observe that the reduction Lemma 4.4 hold in this case, and the proof is basically the same. Therefore, analogous computations give as the following

Proposition 5.1. *Let $n \geq 2$ and V an irreducible finite-dimensional $sl(1|n)$-module. If m is a non-trivial singular vector in the $S(1,n)_+$-module* Tens V *with weight $\bar{\lambda}_m$, then we have one of the following:*

(a) $m = \xi^n \otimes v_n$, $\bar{\lambda}_m = (0, \dots, 0, -k)$ with $k \geq 0$, v_n is a highest weight vector in V with weight $(0, \dots, 0, -k-1)$, and m is uniquely defined by v_n.

(b) $m = \sum_{l=1}^{n} \xi^l \otimes v_l$, $\bar{\lambda}_m = (k, 1, \dots, 1)$ with $k \geq 2$, v_1 is a highest weight vector in V with weight $(k-1, 1, \dots, 1)$, and m is uniquely defined by v_1.

(c) $m = \partial(\xi_* \otimes w) + \sum_{l=1}^{n} \xi^l \otimes v_l$, $\bar{\lambda}_m = (0, \dots, 0)$, w is a highest weight vector in V with weight $(1, \dots, 1)$, and m is uniquely defined by w.

(d) $m = \partial(\xi^n \otimes w) + \sum_{l=1}^{n-1} \xi_{[l,n]-\{l,n\}} \otimes v_l + \xi^n \otimes v_n$, $\bar{\lambda}_m = (0, \dots, 0, -1)$, w is a highest weight vector in V with weight $(1, \dots, 1)$, and m is uniquely defined by w.

Using the above proposition, we have

Theorem 5.2. *Let $L = S(1,n)_+$ $(n \geq 2)$ and F be an irreducible L_0-module with highest weight $\bar{\lambda}_*$. Then the L-modules $\mathrm{Ind}_{L_{\geq 0}}^{L} F$ are irreducible finite continuous modules except for the following cases:*

(a) $\bar{\lambda}_* = (0, \dots, 0, -p), p \geq 0$, where $\mathrm{Ind}_{L_{\geq 0}}^{L} F = \Theta_+^p$ and the image $d^{\#}\Theta_+^{p+1}$ is the only non-trivial proper submodule.

(b) $\bar{\lambda}_* = (q, 1, \ldots, 1), q \geq 1$, *where* $\mathrm{Ind}_{L_{\geq 0}}^{L} F = \Omega_-^q$. *For* $q \geq 2$ *the image* $d\Omega_-^{q-1}$ *is the only non-trivial proper submodule. For* $q = 1$*, the proper submodules are* $\mathrm{Im}(d)$, $\mathrm{Ker}(d)$ *and* $\mathrm{Im}(\alpha)$*, where* α *is the composition*

$$\alpha : \Theta_+^1 \xrightarrow{d^\#} \Theta_+^0 \simeq \Omega_-^0 \xrightarrow{d} \Omega_-^1,$$

and $\mathrm{Ker}(d)$ *is the maximal proper submodule.*

Proof. Similarly to the case of $W(1,n)_+$, the modules $\mathrm{Ind}_{L_{\geq 0}}^{L} F$ are irreducible except when they have a singular vector and the highest weights of such F, when it could happen, are listed in (a), (b), (c) and (d) of the above Proposition 5.1. The weight $(1, \ldots, 1)$ is special here because it is relevant to (b), (c) and (d). There are three types of singular vectors possible in this case. The corresponding module $\mathrm{Ind}(F) = \Omega_-^1$ has three different submodules and all three vectors are present. The same argument as for $W(1,n)_+$-modules allows us easily to conclude that the listed submodules are the only ones and the factors are irreducible. $\square$

Corollary 5.3. *The theorem gives us a description of finite continuous irreducible* $S(1,n)_+$*-modules when* $n \geq 2$*. Such a module is either* $\mathrm{Ind}_{L_{\geq 0}}^{L} F$ *for an irreducible finite-dimensional* L_0*-module* F *where the highest weight of* T *does not belong to the types listed in (a), (b) of the theorem or the factor of an induced module from (a), (b) by the submodule* $\mathrm{Ker}(d)$*.*

Corollary 5.4. *The Lie superalgebras* $S(1,n)_+$ *and* $S(1,n)'_+$ *have the same finite continuous irreducible modules, and they are described by the previous corollary.*

Proof. In order to see that Theorem 5.2 also holds for $S(1,n)'_+$, it is basically enough to see that Proposition 5.1 holds in this case. But, if we check the proof in the classification of singular vectors, we see that the element $\xi_1 \cdots \xi_n \partial_0$ (cf. (5.1)) appears only in the condition (s3) in (4.10) in the special case $g = \xi_1 \cdots \xi_n$ and $j = 0$. If we track the details of the proof, we see that this special constrain only produces equation (4.27) for $n = 2$, but in any case, this equation is not used in the rest of the proof. Therefore, the singular vectors are the same for both Lie superalgebras $S(1,n)'_+$ and $S(1,n)_+$, finishing the proof. $\square$

Now, as in the W_n case, Theorem 5.2 and Corollary 5.4, along with Section 3 and Propositions 2.3, 2.8 and 2.9 give us a complete description of finite irreducible S_n-modules ($n \geq 2$): it is given by Theorem 4.11 in which W_n is replaced by S_n and $gl(1|n)$ is replaced by $sl(1|n)$.

Remark 5.5. Under the standard isomorphism between S_2 and small N=4 conformal superalgebra it is easy to see that our result agrees with the classification given in [4]. Indeed, in [4] (Theorem 6.1) the classification of irreducible modules

was given in terms of parameters Λ and Δ, and these parameters are related to ours as follows,

$$\lambda_1 = -\Delta + \frac{\Lambda}{2}, \tag{5.3}$$

$$\lambda_2 = -\Delta - \frac{\Lambda}{2}. \tag{5.4}$$

Therefore, the case $2\Delta - \Lambda = 0$ ($\Lambda \in \mathbb{Z}_+$) corresponds to the family $\Omega^*_{\Lambda,\alpha}/\mathrm{Ker}\ \widetilde{d}^*$ of rank 4Λ, and the case $2\Delta + \Lambda + 2 = 0$ ($\Lambda \in \mathbb{Z}_+$) corresponds to $\Omega_{\Lambda+1,\alpha}/\mathrm{Ker}\ \widetilde{d}$ of rank $4\Lambda + 8$. Therefore, we have one module of rank 4 that corresponds to $\Omega^*_1/Ker\ \widetilde{d}^*$, and by Remark 4.12, the dual of this module is Ω_0 (Ker is trivial in this case) and (using Proposition 4.3) Ω_0 is the tensor module $Tens(V)$ where V is the trivial representation, therefore it is reducible with a maximal submodule of codimension 1 (over $\mathbb{C}$).

6 Lie conformal superalgebras $S_{n,b}$ and $\widetilde{S}_n$, and their finite irreducible modules

- *Case $S_{n,b}$:*

For any $b \in \mathbb{C}$, $b \neq 0$, we take

$$S(1, n, b) = \{D \in W(1, n) | div(e^{bx}D) = 0\}.$$

This is a formal distribution subalgebra of $W(1, n)$. The associated Lie conformal superalgebra is constructed explicitly as follows. Let $D = \sum_{i=1}^n P_i(\partial, \xi)\partial_i + f(\partial, \xi)$ be an element of W_n. We define the deformed divergence as

$$div_b D = div D + bf.$$

It still satisfies equation 5.2, therefore

$$S_{n,b} = \{D \in W_n | div_b D = 0\}$$

is a subalgebra of W_n, which is simple for $n \geq 2$ and has rank $n2^n$. Since $S_{n,0} = S_n$ has been discussed in the previous section, we can (and will) assume that $b \neq 0$.

If $b \neq 0$, the extended annihilation algebra is given by

$$(Alg(S_{n,b}))^+ = \mathbb{C}ad(\partial_0 - b \sum_{i=1}^n \xi_i \partial_i) \ltimes S(1, n)_+ \simeq CS(1, n)'_+$$

where $CS(1, n)'_+$ is obtained from $S(1, n)'_+$ by enlarging $sl(1, n)$ to $gl(1, n)$ in the 0th-component.

Therefore, the construction of all finite irreducible modules over $S_{n,b}$ is the same as that for W_n, but without twisting by α. Hence, using Theorem 4.11, we have

Theorem 6.1. *The following is a complete list of finite irreducible $S_{n,b}$-modules $(n \geq 2, b \in \mathbb{C}, b \neq 0)$:*

(a) $\mathrm{Tens} V$*, where* V *is a finite-dimensional irreducible* $gl(1|n)$*-module different from* $\Lambda^k(\mathbb{C}^{1|n})^*, k = 1, 2, \ldots$ *and* $\Lambda^k(\mathbb{C}^{1|n})$*,* $k = 0, 1, 2, ...,$

(b) $\Omega_k^*/\mathrm{Ker}\ \widetilde{d}^*, k = 1, 2, \ldots,$ *and the same modules with reversed parity,*

(c) $S_{n,b}$*-modules dual to* (b)*, with* $k > 1$.

- *Case* $\widetilde{S}_n$*:*

Let $n \in \mathbb{Z}_+$ be an even integer. We take

$$\widetilde{S}(1, n) = \{D \in W(1, n) | div((1 + \xi_1 \ldots \xi_n)D) = 0\}.$$

This is a formal distribution subalgebra of $W(1, n)$. The associated Lie conformal superalgebra $\widetilde{S}_n$ is constructed explicitly as follows:

$$\widetilde{S}_n = \{D \in W_n | div((1 + \xi_1 \ldots \xi_n)D) = 0\} = (1 - \xi_1 \ldots \xi_n)S_n.$$

The Lie conformal superalgebra $\widetilde{S}_n$ is simple for $n \geq 2$ and has rank $n2^n$.

The extended annihilation algebra is given by

$$(Alg(\widetilde{S}_n))^+ = \mathbb{C}ad(\partial_0 - \xi_1 \ldots \xi_n \partial_0) \ltimes S(1, n)'_+ \simeq S(1, n)_+.$$

Therefore, the construction of all finite irreducible modules over $\widetilde{S}_n$ is the same as that for S_n, but without the twist by α.

Acknowledgment. C. Boyallian and J. Liberati were supported in part by grants of Conicet, ANPCyT, Fundación Antorchas, Agencia Cba Ciencia, Secyt-UNC and Fomec (Argentina). Special thanks go to MSRI (Berkeley) for the hospitality during our stay there, where part of this work started.

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